Kruskal Dynamics For Radial Geodesics. I

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The total spacetime manifold for a Schwarzschild black hole (BH) is described by the Kruskal coordinates $u = u(r, t)$ and $v = v(r, t)$, where r and t are the conventional Schwarzschild radial and time coordinates respectively. The relationship between r and t for a test particle moving on a radial or non-radial geodesic is well known. Similarly, the expression for the vacuum Schwarzschild derivative for a geodesic, in terms of the constants of motion, is well known. However, the same is not true for the Kruskal coordinates; and, we derive here the expression for the Kruskal derivative for a radial geodesic in terms of the constants of motion. In particular, it is seen that the value of $\mid du/dv \mid (= 1)$ is regular on the Event Horizon of the Black Hole. The *regular nature* of the Kruskal derivative is in sharp contrast with the Schwarzschild derivative, $\mid dt/dr \mid = \infty$, at the Event Horizon. We also explicitly obtain the value of the Kruskal coordinates on the Event Horizon as a function of the constant of motion for a test particle on a radial geodesic.

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I. INTRODUCTION

It is known for more than 80 years that the region exterior to a point mass or the event horizon ($r > r_g = 2m$) of a Schwarzschild Black Hole (BH) can be described by the vacuum Schwarzschild metric [1,2]:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{1}$$

where $g_{tt} = (1 - 2m/r)$, $g_{rr} = -(1 - 2m/r)^{-1}$, $g_{\theta\theta} = -r^2$, and $g_{\phi\phi} = -r^2 \sin^2\theta$. Here, we are working with a spacetime signature of +1, -1, -1, -1 and r has a distinct physical significance as the *invariant circumference radius*. The coordinate time t too has a physical significance as the proper time of a distant inertial observer S_∞. At $r = 2m$, g_{rr} blows up and as $r < 2m$, the g_{tt} and g_{rr} suddenly exchange their signatures though the signatures of $g_{\theta\theta}$ and $g_{\phi\phi}$ remain unchanged. The detail dynamics of a "test particle" in the vacuum external spacetime is well known for a very long time and discussion on it is contained in practically every text book or monograpgh on classical General Theory of Relativity (GTR). One of the key aspects for studying the kinematics of a test particle is the knowledge about the relevant derivative of the spatial coordinate with the temporal one. For instance for any geodesic having angular momentum or not, one knows the details about the behaviour of the Schwarzschild derivative dr/dt or dt/dr. And the fact that $dt/dr = -\infty$ blows up at the Event Horizon restricts the utility of the Schwarzschild dynamics below the EH. This tantamounts to the well known fact that the vacuum Schwarzschild metric fails to describe the spacetime inside $r \le 2m$.

On the other hand, we learnt in 1960 that both the exterior and the interior regions of a BH may be described by a one-piece coordinate system suggested by Kruskal and Szekeres [3,4]. Though, in the intervening 39 years hundreds of articles have been written on Kruskal coordinates, and, most of the treatises on GTR too regularrly deliberate upon the original work of Kruskal and Szekers, the fact remains that sufficient effort has not been made to study the kinematics of a test particle in terms of the Kruskal coordinates so that one could have a better insight and appreciation of the kinematics inside the EH. As a first step towards this direction, in this paper, we would derive expressions for the Kruskal derivative du/dv for a radial geodesic. For the sake of completeness, we shall start from the usual description about the Kruskal coordinates and first derive the exact expression for the value of the Kruskal coordinates on the EH (u_H and v_H) in terms of r and t. We shall show here that u_H and v_H are always non-zero and finite in general. More importantly, we shall explicitly show that unlike the Schwarzschild derivative, the Kruskal derivates are regular on the EH in accordance with the singularity free nature of the Kruskal coordinates.

II. KRUSKAL COORDINATES

For the region exterior to the EH (Sectors I & III), the Kruskal coordinates are defined as follows:

$$u = f_1(r)\cosh\frac{t}{4m}; \qquad v = f_1(r)\sinh\frac{t}{4m}; \quad r \ge 2m \tag{2}$$

where

$$f_1(r) = \pm\left(\frac{r}{2m} - 1\right)^{1/2} e^{r/4m} \tag{3}$$

Here the plus sign corresponds to "our universe" while the negative sign corresponds to the "other universe" [1,2]. The "other universe" is a legitimate mathematical solution of the Schwarzschild problem (irrespective of its observational reality), and is a time reversed mirror image of "our universe".

And for the region interior to the horizon (Sectors II & IV), we have

$$u = f_2(r)\sinh\frac{t}{4m}; \qquad v = f_2(r)\cosh\frac{t}{4m}; \quad r \le 2m \tag{4}$$

where

$$f_2(r) = \pm\left(1 - \frac{r}{2m}\right)^{1/2} e^{r/4m} \tag{5}$$

In terms of u and v, the metric for the entire spacetime is

$$ds^2 = \frac{32M^3}{r} e^{-r/2m}(dv^2 - du^2) - r^2(d\theta^2 + d\phi^2\sin^2\theta) \tag{6}$$

The metric coefficients are regular everywhere except at the intrinsic singularity $r = 0$, as is expected. Since afterall the Kruskal coordinates are defined using r and t, for a proper understanding of the Kruskal dynamics, it is necessary to recall the inter-relationship between the Schwarzschild coordinates for a geodesic.

A. Inter Relation Between Schwarzschild Coordinates

For a test particle on a radial geodesic, the angular momentum is zero, and there is only one conserved quantity, the energy of the particle (per unit rest mass), E, as measured by a distant inertial observer:

$$E \equiv \frac{dt}{ds}(1 - 2m/r) \tag{7}$$

where s is the proper time. For a massless particle like a photon, we have $E = \infty$, otherwise E is finite. For a radial geodesic, the motion of the particle is determined by (see Chandrasekhar, pp. 98) [5]

$$\frac{dr}{ds} = -\sqrt{E^2 - (1 - 2m/r)} \tag{8}$$

and

$$\frac{dt}{ds} = \frac{E}{1 - 2m/r} \tag{9}$$

so that

$$\frac{dt}{dr} = -\frac{E(1 - 2m/r)^{-1}}{\sqrt{E^2 - (1 - 2m/r)}} \tag{10}$$

Clearly as $r \to 2m$, $dt/dr \to -\infty$. On the other hand, we do not expect such irregular behaviour for the Kruskal derivative.

Here note that if the particle is released from rest ($dr/ds = 0$) at $r = r_i$ at $t = 0$, from Eq. (8), it is seen that [5]

$$E^2 = (1 - 2m/r_i) \tag{11}$$

or,

$$r_i/2m = (1 - E^2)^{-1} \tag{12}$$

It is convenient to introduce a (cylic) parameter η through

$$r = \frac{r_i}{2}(1 + \cos\eta) = \frac{2m}{1 - E^2}\cos^2(\eta/2) = r_i \cos^2(\eta/2) \tag{13}$$

Obviously, $\eta = 0$ when $r = r_i$ and at the EH, we have

$$\eta = \eta_H = 2 \arcsin E; \qquad r = 2m \tag{14}$$

Now after some manipulation, Chandrasekhar arrived at the following Eq. involving t and η [5]:

$$\frac{dt}{d\eta} = E\left(\frac{r_i}{2m}\right)^{1/2} \frac{\cos^4(\eta/2)}{\cos^2(\eta/2) - \cos^2(\eta_H/2)} \tag{15}$$

This Eq. can be integrated to find the exact relation between t and r for a radial geodesic (actually, even for non-radial geodesic this Eq. would hold good):

$$\frac{t}{2m} = E\left(\frac{r_i}{2m}\right)^{3/2} [\frac{1}{2}(\eta + \sin\eta) + (1 - E^2)\eta] + \ln\left[\frac{\tan(\eta_H/2) + \tan(\eta/2)}{\tan(\eta_H/2) - \tan(\eta/2)}\right] \tag{16}$$

The above Eq. may also be written without introducing η_H and E explicitly: (see pp. 824 of ref.[1] or pp. 343 of ref.[2]):

$$\frac{t}{2m} = \ln\mid \frac{(r_i/2m - 1)^{1/2} + (r_i/r - 1)^{1/2}}{(r_i/2m - 1)^{1/2} - (r_i/r - 1)^{1/2}} \mid + \left(\frac{r_i}{2m} - 1\right)^{1/2} \left[\eta + \left(\frac{r_i}{4m}\right)(\eta + \sin\eta)\right] \tag{17}$$

We find from Eqs. (16-17) that, as $r \to 2m$ from Sector I, the logarithmic term blows up and $t \to \infty$, which is a well known result. Further Kruskal coordinates envisage that approach to the EH from the Sectors III & IV corresponds to $t = -\infty$.

B. Kruskal Coordinates on the Event Horizon

In Sectors I & III, Kruskal coordinates obey the relation

$$\frac{u}{v} = \coth \frac{t}{4m} \tag{18}$$

And since $r \to 2m$ corresponds to $t \to \pm\infty$, at the EH, we have

$$\frac{u_H}{v_H} = \pm 1; \qquad r = 2m \tag{19}$$

On the other hand, in Sectors II and IV, we see

$$\frac{u}{v} = \tanh \frac{t}{4m} \tag{20}$$

and as $r \to 2m$, $t \to \pm\infty$, we are led to the same Eq. (19). In the same limit, $r \to 2m$ and $t \to \pm\infty$, we find that

$$u_H^2 = v_H^2 \to f_1^2 \exp \frac{t}{2m} \tag{21}$$

It might appear that since $f_1(2m) = f_2(2m) = 0$ on the EH, we would have $u_H = \pm v_H = 0$. But this is incorrect because the temporal part of u and v tend to blow up much more rapidly on the EH. And one has to carefully obtain the actual values of u_H and v_H by working out appropriate limits

To do so we introduce a new variable

$$z = r_i/2m - 1 = \frac{E^2}{1 - E^2} \tag{22}$$

and, let, in the vicinity of the EH,

$$r/2m = 1 + \epsilon; \qquad \epsilon \to 0 \tag{23}$$

so that

$$f_i^2(2m) \to \epsilon \tag{24}$$

Then, in the vicinity of the EH, by retaining terms first order in ϵ, we can rewrite Eq.(17) as

$$\frac{t}{2m} = \ln | \frac{z^{1/2} + z^{1/2}\left(1 - \frac{\epsilon r_i}{4mz}\right)}{z^{1/2} - z^{1/2}\left(1 - \frac{\epsilon r_i}{4mz}\right)} | + \left(\frac{r_i}{2m} - 1\right)^{1/2} \left[\eta + \left(\frac{r_i}{4m}\right)(\eta + \sin\eta)\right] \tag{25}$$

As $\epsilon \to 0$, the logarithmic term in the above expression becomes

$$A(t) = \ln | \frac{1 + 1 - \frac{\epsilon r_i}{4mz}}{1 - 1 + \frac{\epsilon r_i}{4mz}} | \to \ln \frac{8mz}{r_i \epsilon} \tag{26}$$

Then, using Eqs. (24) and (26), we find

$$f_1^2 \exp(A) \to \frac{8emz}{r_i} = 4e(1 - 2m/r_i) = 4eE^2 \tag{27}$$

Now considering the other terms in the expression for $t/2m$ in Eq.(25), we find that, in this limit,

$$u_H^2 = v_H^2 = 4e(1 - 2m/r_i) \exp\left\{\left(\frac{r_i}{2m} - 1\right)^{1/2} \left[\eta_H + \left(\frac{r_i}{4m}\right)(\eta_H + \sin\eta_H)\right]\right\} \tag{28}$$

In terms of E, we have

$$u_H^2 = v_H^2 = 4eE^2 \exp\ E\left[\eta_H + \frac{\eta_H + \sin\eta_H}{2\sqrt{1 - E^2}}\right] \tag{29}$$

One would have $u_H = v_H = 0$ if $E = 0$ or, if the test particle is injected *from rest* right at the EH. Clearly, this is unphysical, and thus we see that u_H and v_H are non-zero. Further, for a finite value of $r_i/2m$ or for $E < 1$, they are finite too. The finiteness of u and v at the EH is physically appealing because u and v are expected to be completely regular at the EH. However for $r_i/2m = \infty$ or $E = 1$, we find $u_H^2 = v_H^2 = \infty$.

On the other hand, since $r = r_i$ at $t = 0$, by using the definition of u and v, we find that the initial values of

$$u^2 = u_i^2 = (r_i/2m - 1) = \frac{E^2}{1 - E^2} \tag{30}$$

and

$$v^2 = v_i^2 = 0 \tag{31}$$

III. KRUSKAL DERIVATIVE: A DIRECT APPROACH

Having shown that u_H and v_H are, in general, non-zero, we are now in a position to evaluate the Kruskal derivative, the key ingredient for studying the Kruskal dynamics for a radial geodesic. We first confine ourselves to Sector I. By differentiating $f_1(r)$ (Eq.[3]) with r we obtain

$$\frac{df_1}{dr} = \frac{\pm r}{2m} \frac{e^{r/4m}}{4m} (r/2m - 1)^{-1/2} \tag{32}$$

Then by directly differentiating Eq.(2) by r, we find that irrespective of the sign of df_1/dr, we will have

$$\frac{du}{dr} = \frac{df_1}{dr} \cosh\frac{t}{4m} + \frac{f_1}{4m} \sinh\frac{t}{4m} \frac{dt}{dr} \tag{33}$$

Interestingly, in all the sectors, we obtain the same functional form of du/dr. Using Eqs.(2) and (4) in the foregoing Eq., we see that

$$\frac{du}{dr} = \frac{u}{4m}(1 - 2m/r)^{-1} + \frac{v}{4m} \frac{dt}{dr} \tag{34}$$

On the other hand by differentiating Eqs. (4) and (5), we find that

$$\frac{df_2}{dr} = \frac{\mp r}{2m} \frac{e^{r/4m}}{4m} (r/2m - 1)^{-1/2} \tag{35}$$

and

$$\frac{du}{dr} = \frac{df_2}{dr} \sinh\frac{t}{4m} + \frac{f_2}{4m} \cosh\frac{t}{4m} \frac{dt}{dr} \tag{36}$$

And by using Eqs. (4) and (35) into the foregoing Eq., we obtain the *same expression* (34) for du/dr in Sectors II & IV. Further, using Eq.(10) in (34), we obtain the ultimate expression for

$$\frac{du}{dr} = \frac{(1 - 2m/r)^{-1}}{4m} \left[u - \frac{vE}{\sqrt{E^2 - 1 + 2m/r}} \right] \tag{37}$$

valid in all the sectors. Similarly, we obtain the ultimate functional form of dv/dr which is valid for *all the sectors*:

$$\frac{dv}{dr} = \frac{(1 - 2m/r)^{-1}}{4m} \left[v - \frac{uE}{\sqrt{E^2 - 1 + 2m/r}} \right] \tag{38}$$

And, the general value of du/dv in any Sector is obtained by dividing Eq.(37) with (38):

$$\frac{du}{dv} = \frac{u - \frac{vE}{\sqrt{E^2-1+2m/r}}}{v - \frac{uE}{\sqrt{E^2-1+2m/r}}} \tag{39}$$

A. Kruskal Derivative at the Event Horizon

Since u and v are expected to be differentiable smooth continuous (singularity free) functions everywhere except at $r=0$, and also since the "other universe" is a mirror image of "our universe", we expect that the value of du/dv for any given r must be the same, except for a probable difference in the signature, in both the universes. The meaningful way to find the value of du/dr at the EH will be to concentrate on the Sectors II & IV for which $u_H = -v_H$

$$\frac{du}{dv} \to \frac{u-v}{v-u} = \frac{2u_H}{2v_H} = -1; \qquad r = 2m \tag{40}$$

The Eq. (39) for the Kruskal redivative, however, tends to yield a "0/0" form at $r=2m$ for Sectors I & III having $u_H = v_H$. But as mentioned above, we expect this 0/0 form to acquire the value $du/dv = +1$ because these Scetors are the mirror images of Sectors II & IV. Otherwise the whole idea of having an extended time symmetric Schwarzschild manifold would be inconsistent. Thus, in general, we must have

$$du/dv = \pm 1; \qquad r = 2m \tag{41}$$

The fact that we must have $du/dv = +1$ for the Sectors I & III can be reconfirmed in the limiting case of $u_H^2 = v_H^2 = \infty$ for $E=1$ or $u_H = v_H = 0$ for the (unphysical case) $E=0$ directly by using L' Hospital's theorem.

Note that, by this rule, we can write,

$$\lim_{u\to 0(\infty), v\to 0(\infty)} \frac{u}{v} = \lim_{u\to 0(\infty), v\to 0(\infty)} \frac{du/dr}{dv/dr} \tag{42}$$

In any case, from Eq. (19), we already know that $u/v = \pm 1$ at $r=2m$. Then we can rearrange the foregoing Eq. as

$$\lim_{u\to 0(\infty), v\to 0(\infty)} \frac{du/dr}{dv/dr} = \pm 1 \tag{43}$$

or,

$$\lim_{r\to 2m} \frac{du}{dv} = \pm 1 \tag{44}$$

IV. A DIFFERENT ROUTE

It may be of some interest to rederive the limiting value of du/dv by using other generic relationships between u and v. As before, to avoid 0/0 forms, we work with Sectors III & IV. In particular, in Sector III, we have

$$\frac{u}{v} = \coth \frac{t}{4m} \tag{45}$$

By differentiating this equation w.r.t. v, we obtain

$$\frac{1}{v}\frac{du}{dv} - \frac{u}{v^2} = -\frac{1}{4m}\frac{1}{\sinh^2(t/4m)}\ \frac{dt}{dv} \tag{46}$$

By recalling that $\sinh(t/4m) = v/f_1$, we rewrite the above Eq. as

$$\frac{du}{dv} - \frac{u}{v} = \frac{-f_1^2}{4m}\frac{1}{v}\frac{dt}{dv} \tag{47}$$

Now, from Eqs. (10) and (38), note that

$$\frac{dt}{dv} = \frac{dt}{dr}\frac{dr}{dv} = -\frac{4mE}{v\sqrt{E^2 - 1 + 2m/r} - uE} \tag{48}$$

And the limiting value of

$$\frac{dt}{dv} \to \frac{4m}{u-v} = \frac{4m}{u_H + u_H} = \frac{2m}{u_H} \qquad r \to 2m \tag{49}$$

And since $f_1(2m) = 0$, we find from Eq. (47) that

$$\frac{du}{dv} - \frac{u}{v} = 0; \qquad r = 2m \tag{50}$$

Or,

$$\frac{du}{dv} = \frac{u}{v} = \frac{u_H}{-u_H} = -1; \qquad r = 2m\ (Sector\ III + IV) \tag{51}$$

Similarly, for the sake of overall consistency, in Sectors, I & III, we must have $du/dv = +1$ at $r = 2m$.

V. A DIFFERENT CONSIDERATION

Actually we could have obtained the above derived unique result in a relatively simpler manner by differentiating the Global Eq.

$$u^2 - v^2 = (r/2m - 1)e^{r/2m} \tag{52}$$

w.r.t. v :

$$2u\, \frac{du}{dv} - 2v = \frac{r}{4m^2} e^{r/2m}\, \frac{dr}{dv} \tag{53}$$

First let us note from Eq.(37) that in Sectors II & IV, the limiting value of

$$\frac{dr}{dv} \to \frac{4m(1-2m/r)}{v-u} \to \frac{2m(1-2m/r)}{v_H} = 0; \qquad r \to 2m \tag{54}$$

Then, by using this above Eq. in (53), we find

$$u_H\, \frac{du}{dv} = v_H; \qquad r \to 2m \tag{55}$$

so that

$$\frac{du}{dv} = \frac{v_H}{-v_H} = -1; \qquad r \to 2m \tag{56}$$

VI. CONCLUSIONS

The Kruskal coordinates were found way back in 1960, and in the present paper, we have worked out some aspects of the kinematics of a test particle following a radial Kruskal geodesic. To attain this we used, for the first time, the precise value of u_H and v_H as a function of the initial conditions of the problem r_i, m or E. It is clearly found that $u_H^2 = v_H^2$ is non-zero in general.

We then proceeded to obtain the expression for the Kruskal derivative in terms of m, E and r. We found that the *Kruskal derivative is regular at the EH unlike the Scharzschild deivative(s)* where $dt/dr = -\infty$ at the EH.

In particular $du/dv = +1$ at the EH if we consider the "other universe"whose existence is suggested by the full Kruskal manifold, and which is a time reversed version of "our universe". But, if we move to the "our universe", the expected value of $du/dv = 1$ at the EH.

The *regular nature of the Kruskal derivative is in keeping with the notion that Kruskal coorinates are free of singularities at the EH.*

In a subsequent paper, we shall find out other important features of the Kruskal dynamics vis-a-vis the well known Schwarzschild dynamics.

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